Numerical Method for Singularly Perturbed Boundary Value Problems for Third Order Ordinary Differential Equations

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1 **Introduction to Singular Perturbation Problems**

2 **Computational method**

3 **Conclusions**

Introduction to Singular Perturbation Problems(SPPs)

- The birth of the singular perturbations was introduced by Prandtl at the Third International Congress of Mathematicians in Heidelberg in 1904 and it was reported in the proceedings of the conference .

- Many practical problems, such as the mathematical boundary layer theory or approximation of solutions of various problems are described by differential equations involving large or small parameters.

Let P_ε denote the original problem and u_ε be its solution.
Let P_0 denote the reduced problem of P_ε (setting $\varepsilon = 0$ in P_ε) and u_0 be its solution. Then the problem P_ε is called a **Singularly Perturbed Problem (SPP)** if and only if u_ε does not converge uniformly to u_0 in the entire domain of the definition of the problem. Otherwise the problem is called **Regularly Perturbed Problem (RPP)**.

Example 1.1 (Singular Perturbation Problem)

$$P_\varepsilon : \quad \begin{cases} \varepsilon u_\varepsilon'(x) = -u_\varepsilon(x), \ x \in (0, 1], \\ u_\varepsilon(0) = 1, \quad 0 < \varepsilon \ll 1. \end{cases}$$

$$P_0 : \quad u_0(x) = 0, \ x \in [0, 1],$$

The exact solution of P_ε is given by

$$u_\varepsilon(x) = e^{-x/\varepsilon}.$$

Note that,

$$\lim_{\varepsilon \to 0} \lim_{x \to 0} u_\varepsilon(x) = 1 \neq 0 = \lim_{x \to 0} \lim_{\varepsilon \to 0} u_\varepsilon(x).$$

That is, $u_\varepsilon(x)$ does not converge uniformly to the reduced problem solution on $[0, 1]$. The solution changes very rapidly near the neighborhood of $x = 0$. This neighborhood is called a **boundary layer**.

- A singular perturbation problem is said to be of convection-diffusion type, if the order of the differential equation is reduced by one when the perturbation parameter is set equal to zero.
- If the order reduces by two, it is known as a reaction-diffusion type problem.

Computational method

Using the Boundary Value Technique (BVT) of [3] and basic idea of [2], we have suggested a new computational method. This method makes use of the zero order asymptotic expansion approximation, BVT and Shooting method to obtain a numerical solution for the derivative of SPBVP for third order ODEs of convection-diffusion type of the form:

$$\varepsilon y'''(x) + a(x)y''(x) - b(x)y'(x) - c(x)y(x) = f(x),\ x \in \Omega, \quad (1)$$

$$y(0) = p, \quad -y''(0) = q, \quad y'(1) - y''(1) = r. \quad (2)$$

where $y \in C^{(3)}(\Omega) \cap C^{(2)}(\bar{\Omega})$, $0 < \varepsilon \ll 1$, $a(x), b(x), c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the following conditions .

$$a(x) \geq \alpha, \alpha > 0,$$
$$b(x) > 0,$$
$$0 \geq c(x) \geq -\gamma, \gamma > 0, \quad \alpha > \gamma.$$

The SPBVP (1)-(2) can be transformed into an equivalent weakly coupled system of the form:

$$
\begin{cases}
P_1 \bar{y}(x) \equiv -y_1'(x) + y_2(x) = 0, \ x \in \Omega^0 = (0, 1], \\
P_2 \bar{y}(x) \equiv \varepsilon y_2''(x) + a(x) y_2'(x) - b(x) y_2(x) \\
-c(x) y_1(x) = f(x), \ x \in \Omega = (0, 1), \\
y_1(0) = p, \quad -y_2'(0) = q, \quad y_2(1) - y_2'(1) = r,
\end{cases}
\tag{3}
$$

where $\bar{y} = (y_1, y_2)^T$, $y_1 \in C^{(3)}(\Omega) \cap C^{(2)}(\bar{\Omega})$, $y_2 \in C^{(2)}(\Omega) \cap C^{(1)}(\bar{\Omega})$, the functions $a(x), b(x), c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the same conditions given above.

Analytical Results

Theorem 2.1

(Maximum Principle).*Consider the BVP (3). Let $y_1(0) \geq 0$, $y_2'(0) \leq 0$ and $y_2(1) - y_2'(1) \geq 0$. Then $P_1\bar{y}(x) \leq 0$, for $x \in \Omega^0$ and $P_2\bar{y}(x) \leq 0$, for $x \in \Omega$, implies that $\bar{y}(x) \geq 0, \forall x \in \bar{\Omega}$.*

Lemma 2.2

(Stability Result).*If $\bar{y}(x)$ is the solution of the BVP (3) then*

$$||\bar{y}(x)|| \leq C \max\{|y_1(0)|, |y_2'(0)|, |y_2(1) - y_2'(1)|, \max_{x \in \Omega^0} |P_1\bar{y}(x)|,$$

$$\max_{x \in \Omega} |P_2\bar{y}(x)|\}, \, \forall x \in \bar{\Omega}.$$

Asymptotic Expansion Approximation

We look for an asymptotic expansion solution of the BVP (3) in the form

$$\bar{y}(x, \varepsilon) = (\bar{u}_0(x) + \bar{v}_0(x)) + \varepsilon(\bar{u}_1(x) + \bar{v}_1(x)) + \cdots$$

By the method of stretching variable [1] one can obtain a zero order asymptotic approximation as $\bar{y}_{as}(x) = \bar{u}_0(x) + \bar{v}_0(x)$, where $\bar{u}_0(x) = (u_{01}(x), u_{02}(x))^T$ is the solution of the reduced problem of the BVP (3) given by

$$\begin{cases} -u'_{0_1}(x) + u_{0_2}(x) = 0, \\ a(x)u'_{0_2}(x) - b(x)u_{0_2}(x) - c(x)u_{0_1}(x) = f(x), \\ u_{0_1}(0) = p, \quad u_{0_2}(1) - u'_{0_2}(1) = r, \end{cases} \tag{4}$$

and $\bar{v}_0(x) = (v_{0_1}(x), v_{0_2}(x))^T$ is a layer correction term satisfies

$$
\begin{cases}
-v'_{0_1}(x) + v_{0_2}(x) = 0, \\
\varepsilon v''_{0_2}(x) + a(0)v'_{0_2}(x) = 0, \\
v_{0_1}(0) = -(\varepsilon/a(0))v_{0_2}(0), \\
v'_{0_2}(0) = -(q + u'_{0_2}(0)), \; v_{0_2}(1) = \exp(-(a(0)/\varepsilon))v_{0_2}(0)
\end{cases}
\tag{5}
$$

and this $\bar{v}_0(x)$ given by

$$
\begin{cases}
v_{0_1}(x) = -(\varepsilon/a(0))(q + u'_{0_2}(0)\exp(-(a(0)/\varepsilon))(x), \\
v_{0_2}(x) = (q + u'_{0_2}(0))\exp(-(a(0)/\varepsilon))(x).
\end{cases}
\tag{6}
$$

The following theorem gives the bound for the difference between the solution of the BVP (3) and its zero order asymptotic expansion approximation.

Theorem 2.3

The zero order asymptotic approximation $\bar{y}_{as} = \bar{u}_0(x) + \bar{v}_0(x)$ of the solution $\bar{y}(x)$ of the BVP (3) defined by (4)-(6) satisfies the inequality

$$||\bar{y}(x) - \bar{y}_{as}(x)|| \leq C\varepsilon, \;\; \forall x \in \bar{\Omega}.$$

Corollary 2.4

If $y_1(x)$ is the solution of the BVP (3) and $u_{0_1}(x)$ is the solution of the problem (4) then $|y_1(x) - u_{0_1}(x)| \leq C\varepsilon, \; \forall x \in \bar{\Omega}.$

Some analytical and numerical results for second order SPBVP

Consider the auxiliary second order SPBVP

$$Ly_2^\star(x) \equiv \varepsilon y_2^{\star''}(x) + a(x)y_2^{\star'}(x) - b(x)y_2^\star(x)$$
$$= f(x) + c(x)u_{0_1}(x), \ x \in \Omega, \tag{7}$$
$$B_0 y_2^\star(0) \equiv -y_2^{\star'}(0) = q, \quad B_1 y_2^\star(1) \equiv y_2^\star(1) - y_2^{\star'}(1) = r, \tag{8}$$

where $u_{0_1}(x)$ is defined as in (4), $a(x)$, $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth and $a(x) \geq \alpha$ and $b(x) > 0$, $0 \geq c(x) \geq -\gamma, \gamma > 0$.

Analytical Results

Theorem 2.5

(Maximum Principle).*Consider the BVP (7)-(8). Let $y_2^\star(x)$ be a smooth function satisfying $B_0 y_2^\star(0) \geq 0$, $B_1 y_2^\star(1) \geq 0$ and $L y_2^\star(x) \leq 0$ for $x \in \Omega$. Then, $y_2^\star(x) \geq 0, \forall x \in \bar{\Omega}$.*

Lemma 2.6

(Stability Result). *If $y_2^\star(x)$ is the solution of the BVP (7)-(8) then*

$$|y_2^\star(x)| \leq C \max\{|B_0 y_2^\star(0)|, |B_1 y_2^\star(1)|, \max_{x \in \Omega} |L y_2^\star(x)|\}, \forall x \in \bar{\Omega}.$$

Theorem 2.7

If $\bar{y}(x)$ and $y_2^\star(x)$ are solutions of the BVP (3) and (7)-(8) respectively, then $|y_2(x) - y_2^\star(x)| \leq C\varepsilon, \forall x \in \bar{\Omega}.$

Description of the method

Step 1: An asymptotic approximation is derived for the solution of (3) which is given by (4)-(5).

Step 2: The first component of the solution $\bar{y}(x)$ of the BVP (3), namely y_1 is approximated by the first component of the solution of the reduced problem namely u_{0_1} given by (4). Then replacing y_1 appearing in the second equation of (3) by u_{0_1} and taking the same boundary values, one gets the auxiliary SPBVP (7)-(8). The solution of this problem is taken as an approximation to y_2 which is the second equation of (3) which has to be solved.

Step 3: In order to solve the auxiliary second order problem (7)-(8) numerically, we divide the interval $[0, 1]$ into two subintervals $[0, \tau]$ and $[\tau, 1]$ called inner and outer region respectively, where $\tau = \min\{\frac{1}{2}, \frac{\varepsilon}{\alpha} \ln N\}$.

Then, from the BVP (7)-(8) two problems namely inner region problem and outer region problem are derived. To find the boundary condition at $x = \tau$, a zero order asymptotic expansion is used.
The inner region problem for (7)-(8) is given by

$$
\begin{cases}
\varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), x \in (0, \tau), \\
-y_2'(0) = q, \ y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) = \bar{r}.
\end{cases}
$$
(9)

The outer region problem for (7)-(8) is given by

$$
\begin{cases}
\varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = f(x) + c(x)u_{0_1}(x), x \in (\tau, 1), \\
y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) = \bar{r}, y_2(1) - y_2'(1) = r,
\end{cases}
$$
(10)

Numerical Schemes

Inner region problem:
Step 4: The inner region problem is solved by the Shooting method.
Here, Shooting method in the sense that the BVP (9) is replaced by the
IVP (11) in the interval $[0, \tau]$.
Consider the following IVP:

$$\begin{cases} \varepsilon \tilde{y}_2''(x) + a(x)\tilde{y}_2'(x) - b(x)\tilde{y}_2(x) = f(x) + c(x)u_{0_1}(x), \ x \in (0, \tau], \\ \tilde{y}_2(0) = \bar{q} = u_{0_2}(0) + v_{0_2}(0), -\tilde{y}_2'(0) = q. \end{cases} \tag{11}$$

This IVP is equivalent to the system

$$P^*\bar{y}^* = \begin{cases} P_1^*\bar{y}^* = -y_1^{*\prime}(x) + y_2^*(x) = 0, \\ P_2^*\bar{y}^* = \varepsilon y_2^{*\prime}(x) + a(x)y_2^*(x) - b(x)y_1^*(x) = f^*(x), \ x \in (0, \tau], \\ y_1^*(0) = \bar{q}, -y_2^*(0) = q, \end{cases} \tag{12}$$

where $f^*(x) = f(x) + c(x)u_{0_1}(x)$, $\bar{y}^* = (y_1^*, y_2^*)^T$, $a(x) \geq \alpha$, $\alpha > 0$, $b(x) > 0$ and $0 \geq c(x) \geq -\gamma$, $\gamma > 0$.
Applying the Euler's Finite Difference scheme on (12), we get

$$
\begin{cases}
P_1^{*N/2}\bar{y}_i^* = -D^- y_{1,i}^* + y_{2,i}^* = 0, \\
P_2^{*N/2}\bar{y}_i^* = \varepsilon D^- y_{2,i}^* + a(x_i)y_{2,i}^* - b(x_i)y_{1,i}^* = f^*(x_i), \ 1 \leq i \leq N/2, \\
y_{1,0}^* = \bar{q}, -y_{2,0}^* = q,
\end{cases}
\tag{13}
$$

where, $D^- y_{j,i}^* = (y_{j,i}^* - y_{j,i-1}^*)/h_1$, $h_1 = \dfrac{2\tau}{N}$, $x_i = x_{i-1} + ih_1$,

$j = 1, 2$. Here, τ is the transition parameter given by
$\tau = \min\{\dfrac{1}{2}, \dfrac{\varepsilon}{\alpha} \ln N\}$. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Outer region problem

Step 5: The outer region problem given in (10) is solved by the Classical Finite Difference (CFD) scheme. Appling the CFD scheme on (10), we get

$$
\begin{cases}
L^{N/2}y_{2,i} := \varepsilon\delta^2 y_{2,i} + a(x_i)D^+ y_{2,i} - b(x_i)y_{2,i} \\
= f(x_i) + c(x_i)u_{0_1}(x_i), 1 \le i \le N/2 - 1, \\
B_0^{N/2}y_{2,0} = \bar{r}, B_1^{N/2}y_{2,N} = y_{2,N/2} - (y_{2,N/2} - y_{2,N/2-1})/h_2 = r,
\end{cases}
\tag{14}
$$

where, $D^+ y_{2,i} = (y_{2,i+1} - y_{2,i})/h_2, \quad \delta^2 y_{2,i} = (y_{2,i+1} - 2y_{2,i} + y_{2,i-1})/h_2^2,$

$$
x_i = x_{i-1} + ih_2, \quad \text{and} \quad h_2 = \frac{2(1-\tau)}{N}.
$$

Here, τ is defined as mentioned above. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Step 6:

After solving both the inner region and outer region problems, we combine their solutions to obtain an approximate solution y_2 for the derivative of the original problem (1)-(2) over the interval $\bar{\Omega}$.

Error Estimates

Inner region problem

Theorem 2.8

Let $\bar{y}^* = (y_1^*, y_2^*)^T$ and $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be respectively, the solutions of (12) and (13). Then

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq CN^{-1} \ln N \quad \text{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}$$

Theorem 2.9

Let $\bar{y}^* = (y_1^*, y_2^*)^T$ and $\bar{y}^{*1} = (y_1^{*1}, y_2^{*1})^T$ be respectively, the solutions of the IVP

$$\begin{cases} y_1^{*'} - y_2^* = 0, \\ \varepsilon y_2^{*'} + a(x)y_2^* - b(x)y_1^* = f(x) + c(x)u_{0_1}, & x \in \Omega, \\ y_1^*(0) = \alpha, \quad y_2^*(0) = \beta. \end{cases} \quad (15)$$

and

$$\begin{cases} y_1^{1*'} - y_2^{1*} = 0, \\ \varepsilon y_2^{1*'} + a(x)y_2^{1*} - b(x)y_1^{1*} = f(x) + c(x)u_{0_1}, & x \in \Omega, \\ y_1^{1*}(0) = \alpha + O(\varepsilon), \quad y_2^{1*}(0) = \beta. \end{cases} \quad (16)$$

then, $\quad ||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\varepsilon.$

Theorem 2.10

Let $\bar{y}^* = (y_1^*, y_2^*)^T$ be the solution of the IVP (15). Further, let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (16) after applying the Euler's Finite Difference scheme as given in (13). Then,

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq C\varepsilon + CN^{-1} \ln N, \quad \text{for} \quad 0 \leq i \leq N/2 \quad \text{and} \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

The BVP (7)-(8) is equivalent to the following IVP

$$\begin{cases} \varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = f^*(x), x \in \Omega, \\ y_2(0) = q^*, \ y_2'(0) = -q. \end{cases} \tag{17}$$

where q^* is the exact value of the solution of the BVP (7)-(8) at $x = 0$. Because of uniqueness of the solutions of the IVP (17) and the BVP (7)-(8), we have the following result on the error estimate for the inner region problem.

Theorem 2.11

Let $y_2^\star(x_i)$ be the solution of the BVP (7)-(8). Further, let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (13). Then,

$$|y_2^\star(x_i) - y_{1,i}^*| \leq C\varepsilon + CN^{-1}\ln N, \quad for \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Theorem 2.12

Let $\bar{y}(x)$ be the solution of the BVP (3) and let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (13). Then,

$$|y_2(x_i) - y_{1,i}^*| \leq C\varepsilon + CN^{-1}\ln N, \quad for \ 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Outer region

Theorem 2.13

Let $y_2(x_i)$ be the solution of the BVP (10) and $y_{2,i}$ be its numerical solution given by (14).Then,

$$|y_2(x_i) - y_{2,i}| \leq CN^{-1} \ln N, \quad for \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Theorem 2.14

Let $y_2^\star(x_i)$ be the solution of the BVP (7)-(8) and $y_{2,i}$ be the numerical solution of the BVP (10) after applying the CFD scheme as given in (14).Then,

$$|y_2^\star(x_i) - y_{2,i}| \leq C\varepsilon + CN^{-1} \ln N, \quad for \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Theorem 2.15

Let $\bar{y}(x)$ be the solution of the BVP (3) and $y_{2,i}$ be the numerical approximation obtained for $y_2(x_i)$ for the BVP (10) after applying the CFD scheme as given in (14). Then,

$$|y_2(x_i) - y_{2,i}| \leq C\varepsilon + CN^{-1} \ln N, \quad \text{for} \quad 0 \leq i \leq N/2, \, x_i \in \bar{\Omega}_\tau^{N/2}.$$

Numerical Illustrations

We present one example to illustrate the theoretical results for the BVP (1)-(2). Let $\bar{y}_{2,i}^{2N}$ be the piecewise linear interpolants of the numerical solution $y_{2,i}^{N}$ on the mesh Ω^{2N}, where N, $2N$ are the number of mesh points. For a finite set of values of $\varepsilon = \{2^{-13}, \cdots, 2^{-19}\}$, we compute the maximum point-wise errors

$$D_{\varepsilon}^{N} = ||y_{2,i}^{N} - \bar{y}_{2,i}^{2N}||_{\Omega^{N}}, \quad D^{N} = \max_{\varepsilon} D_{\varepsilon}^{N}.$$

From these quantities the order of convergence are computed from

$$p^{*} = \min_{N} p^{N}, \quad \text{where,} \quad p^{N} = log_{2} \left\{ \frac{D^{N}}{D^{2N}} \right\}.$$

Example 2.16

Consider the BVP

$$\varepsilon y'''(x) + (2x+3)^2 y''(x) - 2(2x+3)y'(x) + y(x) = 6x,$$
$$y(0) = 0, \quad y''(0) = -1, \quad y'(1) - y''(1) = 1.$$

The numerical results of the problem is presented in the following table.

Table: *Numerical results for y_2-Maximum pointwise errors D_ε^N, D^N and p^N of Example 2.16.*

			Number of mesh points N			
ε	128	256	512	1024	2048	4096
			Boundary layer region			
2^{-13}	7.8605e-011	5.3932e-011	3.5654e-011	2.1894e-011	1.2857e-011	7.2664e-012
2^{-14}	1.9651e-011	1.3483e-011	8.9138e-012	5.4738e-012	3.2148e-012	1.8165e-012
2^{-15}	4.9127e-012	3.3706e-012	2.2284e-012	1.3678e-012	8.0336e-013	4.5430e-013
2^{-16}	1.2284e-012	8.4244e-013	5.5689e-013	4.1722e-013	4.1722e-013	1.1369e-013
2^{-17}	3.0709e-013	2.1072e-013	1.3922e-013	1.0281e-013	4.9960e-014	4.2100e-013
2^{-18}	4.2100e-013	4.2100e-013	4.2100e-013	4.2100e-013	4.2100e-013	4.2477e-013
2^{-19}	4.2477e-013	4.2477e-013	4.2477e-013	4.2477e-013	4.2477e-013	1.5543e-015
			Outer region			
2^{-13}	1.9122e-003	8.9488e-004	4.3219e-004	2.1262e-004	1.0583e-004	5.3223e-005
2^{-14}	1.9111e-003	8.9401e-004	4.3148e-004	2.1200e-004	1.0525e-004	5.2634e-005
2^{-15}	1.9105e-003	8.9358e-004	4.3112e-004	2.1170e-004	1.0497e-004	5.2359e-005
2^{-16}	1.9102e-003	8.9336e-004	4.3095e-004	2.1154e-004	1.0483e-004	5.2225e-005
2^{-17}	1.9101e-003	8.9325e-004	4.3086e-004	2.1147e-004	1.0476e-004	5.2160e-005
2^{-18}	1.9100e-003	8.9320e-004	4.3082e-004	2.1143e-004	1.0473e-004	5.2127e-005
2^{-19}	1.9100e-003	8.9317e-004	4.3080e-004	2.1141e-004	1.0471e-004	5.2111e-005
D^N	1.9122e-003	8.9488e-004	4.3219e-004	2.1262e-004	1.0583e-004	5.3223e-005
p^N	1.0955e+000	1.0500e+000	1.0234e+000	1.0065e+000	p^* 9.9163e-001	
			The order of convergence= 9.9163e-001			

Conclusions

- In [3], both inner and outer region problems are BVPs, whereas in our case the inner region problem is an IVP and the outer region problem is a BVP.

- Though the present method yields almost the first order of convergence as given in [3], it produces very good reduction on the maximum-pointwise error especially in layer region compared with [3].

- The main advantage of this method is that due to decoupling the system, the size of the matrix to be inverted is reduced from $2N - 1$ to $N - 1$. This results in a good reduction of the computation time.

References

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Thank You